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LIMITATIONS OF WORLD TECHNOLOGICAL PROGRESS: MODELING AND VERIFICATION METHODOLOGY

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Abstract. The article is devoted to the modeling of limited technological progress and the methodology of testing this hypothesis.
The concept of the potential for increasing the efficiency of production factors - the ratio of absolute reserve of efficiency improvement to its achieved level - is the basis of original theoretical model. This indicator acts as an argument of exponential function, which has properties similar to the properties of the potential itself. A differential equation in which the potential function shrinks at a constant rate is an elementary version of the model of limited technological progress. A "megawave" that begins in the infinitely distant past and ends in the infinitely distant future is the solution of this equation in the "potential - pace of technological progress" coordinate system. According to the authors, such a solution does not contradict the history of technological development.
The Cobb–Douglas function has been chosen as the initial function for building the econometric model. Based on it, a linear model of long-term economic growth, consisting of second-order growth equations in successive short-term periods, is built.
It is shown that the parameters of short-term equations can be determined by an iterative procedure using the method of dummy variables. At the initial stage of calculations, the years in which the condition of constant effect of scale of production is violated are determined. These years are chosen as moments of shift in the short-term function of output dynamics. Next, null hypotheses are formulated and Student's t-tests are applied for obtained parameters. According to the results of testing these hypotheses, it is determined for which adjacent periods a certain parameter should be the same. After that, the calculations are repeated. If it turns out that in each subsequent period the pace of technological progress acceleration decreases, then this can be considered a strong argument in favor of the hypothesis of its limitation.
Several approaches to the generalization of this econometric model for the study of several countries are proposed.

Keywords: waves of technological progress, production factors, efficiency potential, Cobb–Douglas function, world economy

1. INTRODUCTION

1.1. Problem definition

Scientific and technical revolution began in the middle of the 20th century, but it still remains the leading factor on which various economic, social and political processes depend. The fate of humanity largely depends on the prospects of technological progress. Together with historical and geographical features and population size, the level and pace of scientific and technical development determine the country's place in the world economy and politics.

Any macroeconomic model describing the state and dynamics of national production must be based on a realistic assessment of possible outcome and pace of technological progress.

One of the "axioms" of the modern worldview consists in the assumption of unlimited future development of science and technology. This idea is extrapolated to macroeconomic aspect of technological progress. It is implicitly assumed that the increase in economic efficiency of production can continue indefinitely, and all limitations are temporary and relative.

According to the authors, such a view should be considered as natural, but, most likely, transient result of the era of rapid scientific and technical development. From a general scientific point of view, any dynamic system eventually faces limitations of the directions and paces of its evolution and even its lifetime. In this regard, it seems quite appropriate to pose a similar problem in the macroeconomic theory of technological progress.

1.2. Analysis of recent research and publications

Given the long history of technological progress modeling, the works containing a detailed coverage of this issue are becoming important.

Thus, the article of Romero (2020) is devoted to a review of models implementing the Schumpeterian approach to economic growth. In the opinion of the author, there are important gaps in this macroeconomic literature, which should become the subject of further research.

A review of recent literature on firm dynamics and innovation can be found in the third chapter of the joint monograph "Macroeconomic Modelling of R&D and Innovation Policies" (Akcigit *et al*., 2022).

A detailed excursion into the history of technological progress modeling is presented in the work of Liu and Liu (2022). The authors propose a dynamic model that combines gradual innovations, technological leaps, endogenous cycles and long-run growth. Their model describes the growth trajectories of two sectors - the research and development (R&D) sector and the goods production sector. It follows from the authors' model that the optimal growth path is cyclical and unique.

In modern studies of the driving forces of technological progress, the classic theory of three factors of production is actively used.

In modern studies of driving forces of technological progress, the classic theory of three factors of production is actively used.

Thus, Jones and Liu (2022) have focused attention on technological changes embodied in the improvement of capital. The authors prove that automation and productivity improvement can ensure balanced growth, which satisfies Uzawa's theorem. Casey and Horii (2022) prove a generalized, multi-factor version of this theorem. The neoclassical model of endogenous growth proposed by the authors includes natural resources and directed technical change.

An alternative to the theory of balanced growth is proposed by de la Fonteijne (2018). In his opinion, the very idea of neutral technological progress is wrong. The author uses a variant of the function with constant elasticity of substitution (CES) of production factors, proposed by Klump *et al.* (2011). In this function, capital can completely replace labor:

$$Y = F(K,L) = Y_0\left(\alpha\left(\frac{K}{K_0}\right)^{\varrho} + (1-\alpha)\left(\frac{L}{L_0}\right)^{\rho}\right)^{\frac{\delta}{\varrho}}, \qquad \varsigma = \frac{1}{1-\varrho}, \tag{1}$$

where Y, K, L are volumes of production, capital and labor; δ is an indicator of scale of production; ς is elasticity of substitution.

In the study of Yeo and Lee (2020), the emphasis is on the second factor of production - labor. The authors have conducted quantitative experiments using the computable general equilibrium (CGE) model. As the researchers note, their analysis has shown that there are limits to productivity growth at the expense of technological innovation. According to the CGE model, this problem can be solved due to more active accumulation of human capital.

An econometric study of demographic aspects of technological progress is presented in the work of Madsen and Strulik (2023).

As in other sections of macroeconomics, the theory of technological progress shows a tendency towards microeconomic substantiation of the proposed models.

In line with this mainstream, the study of Umezuki and Yokoo (2019) relies on the overlapping generations (OLG) model. However, as the authors note, their model is simpler than the corresponding analogues of predecessors, in particular, the Matsuyama model (Matsuyama, 2007). The initial assumption of the model of these authors consists in the idea of a discrete choice of a firm from several production technologies. Mathematically, the Umezuki–Yokoo model reduces into a first-order piecewise linear difference equation, which allows the authors to use the results of neural networks modeling. The model contains one endogenous discontinuity. This leads to strong nonlinearity and generates endogenous cycles in the form of a periodic attractor.

A close approach to modeling technological progress is followed by Gräbner and Hornykewycz (2022). In their proposed model, firms accumulate capabilities for production of heterogeneous consumer goods. This model is used by the authors for theoretical analysis of topological structure of various "product spaces".

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In modern conditions, the use of artificial intelligence in economic research is becoming more and more widespread. Thus, Atashbar and Shi (2023) have applied this approach to the real business cycle (RBC) model. They have considered two scenarios of learning artificial intelligence - deterministic without technological shock and stochastic ones. The developers note that this macromodel can be improved by including additional variables, sectors of the economy or algorithms.

Along with works of theoretical direction, works of applied nature are also presented in modern literature.

Thus, Kyzy (2020) has studied technological progress in 19 developed countries in detail during 1973-2017. His study shows that all the main indicators of technological innovation have a strong positive impact on GDP per capita.

Long-term forecast of the development of the world economy is presented in the study of Fontagné *et al*. (2022). The forecast is based on the Macroeconomic model of the global economy (MaGE) with three factors of production - capital, labor and energy, developed in 2013. The MaGE model is based on a function with constant elasticity of substitution (CES) of production factors. In this model, the neoclassical Cobb–Douglas function is one of the arguments of the CES function.

1.3. Formulation of objectives for the article

As it follows from the review of modern studies of technological progress, there is no question in economic literature about the possibility of the existence of absolute limit for increasing economic efficiency. In this regard, the authors have posed and resolved the following questions:

- what should be the simplest equation that corresponds to the hypothesis of limited technological progress and, at the same time, adequately describes the history of economic development?
- what should be the elementary model of the dynamics of the volume of national (or world) production, which would allow to test the hypothesis of technological progress limitation?
- what can be the method of statistical testing of the hypothesis of limited technological progress?

2. MATERIALS AND METHODS

2.1. Initial idea

The models of exogenous technological progress are the simplest models reflecting qualitative changes in production factors. In these models, the growth of efficiency A is a function of time and does not depend on other economic and socio-political factors:

$$\frac{\mathrm{d}}{\mathrm{dt}}\ln A = \upsilon \quad \Rightarrow, \quad A(t) = A(0)\cdot e^{\upsilon t}, \tag{2), (3}$$

where υ is a constant interpreted as the pace of technological progress.

Of course, the simplistic nature of such models makes them unsuitable for analyzing many problems, including the impact of resource allocation between manufacturing and research and development (R&D) sectors. However, this very simplification helps to trace the most long-term, "historical" properties of technological progress, which is the purpose of this study.

Abstracting from all other factors, except for time factor, allows to formulate key questions:

- firstly, is there a limit to the growth of the efficiency of a separate production factor and the economy as a whole?
- secondly, what consequences will probable presence of such a limit have?

In modern natural and technical sciences, no macroscopic process, which would occur at infinite speed or would have a coefficient of useful action of 100%, is known. It is logical to assume that the same is true for economic processes. No matter how much technologies improve, they cannot provide infinite returns per unit of involved resource, or, what is the same, zero costs per unit of manufactured product. Hence, it is quite logical to assume the existence of absolute limit for increasing economic efficiency.

2.2. Research stages

The proposed study is divided into several stages.

First, the concept of the potential for increasing the efficiency of production factors is introduced and the assumption of a constant rate of its reduction is analyzed. Next, the exponential function of the potential is introduced and the assumption of its constant rate of reduction is analyzed.

At the second stage, the choice of the form of production function and the model of short-term dynamics is substantiated.

At the third stage, the criteria for dividing the studied period into short-term intervals are determined.

The study ends with consideration of the main cases of shifting of the short-term dynamics function.

3. THEORETICAL RESULTS AND DISCUSSION

3.1. Continuous models of limited technological progress

3.1.1. Potential for increasing the efficiency of production factors

Let's introduce the concept of the potential for increasing the efficiency of production factors:

$$p = \frac{A_{max} - A}{A}, \tag{4}$$

where A, A_{max} are current and maximum possible efficiency levels. The value of p shows the maximum relative amount by which efficiency can increase at a certain point in time. At zero initial efficiency, the potential is infinite, and at maximum one - zero:

$$\frac{\mathrm{d}p}{\mathrm{d}A} < 0, \qquad A \to 0 \Rightarrow p \to \infty, \qquad A \to A_{max} \Rightarrow p \to 0\,. \qquad (5), (6), (7)$$

The elementary equation that will describe limited technological progress will have a form similar to the equation of constant rate of technological progress:

$$\frac{\mathrm{d}}{\mathrm{dt}}\ln p = -\nu\,, \qquad (8)$$

where ν is a constant rate of reduction of efficiency potential. A function that has an S-shaped form in the "$t - A$" coordinate system is the solution of this equation:

$$A(t)^{-1} = A(0)^{-1} \cdot e^{-\nu t} + A_{max}^{-1} \cdot (1 - e^{-\nu t}). \qquad (9)$$

In the obtained equation, $A(t)^{-1}$, $A(0)^{-1}$ and A_{max}^{-1} inverse values are coefficients of resource costs per product unit. Thus, in the elementary model of limited technological progress, the current cost factor is the mean value of base and minimum levels. In this model, technological progress is a continuous process that has no definite starting point and will continue indefinitely.

Unfortunately, the elementary equation of limited technological progress shows an overly simplistic picture of historical process. Indeed, it follows that the pace of technological progress must continuously decrease as p potential decreases from ν to 0:

$$\frac{\mathrm{d}}{\mathrm{dt}}\ln p = -\nu \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{dt}}\ln A = \nu \cdot \frac{p}{1+p}. \qquad (10)$$

This creates the effect of the maximum pace of progress in early stages of human development, which, of course, is not true.

This effect is eliminated if we introduce the $V(p)$ function, which has the same economic and mathematical properties as the potential itself:

$$V(p) = e^{\lambda p} - 1 \quad \Rightarrow, \qquad \frac{dV}{dp} > 0, \qquad V(0) = 0, \qquad V(\infty) = \infty, \qquad (11), (12), (13), (14)$$

and replace p potential in original dynamics equation with it:

$$\frac{\mathrm{d}}{\mathrm{dt}}\ln V(p) = -\nu. \qquad (15)$$

As a result, the equation of the pace of technological progress will take the form:

$$\nu^{-1} \cdot \frac{\mathrm{d}}{\mathrm{dt}}\ln A = f(P) = \lambda^{-1} \cdot \frac{1 - e^{-\lambda p}}{1+p}. \qquad (16)$$

In the "$p - (\mathrm{d}\ln A/\mathrm{dt})/\nu$" coordinate system, the curve of normalized pace of technological progress $f(p)$ will have the form of a "megawave" that begins in the infinitely distant past and ends in the infinitely distant future (Figure 1).

Such a "wave-like" change in the pace of technological progress is explained by the fact that at early stage of development ($p > p_0$) the efficiency is very low, and the potential for its increase is very large and therefore practically constant. The A/A_{max} coefficient will play the main role in changing the pace of technological progress. This makes it quite easy to increase current level of efficiency. At late stage of development ($p < p_1$), the $1 - e^{-\lambda p}$ coefficient will play the main role, as a result of which the pace of technological progress will begin to decrease. In $p_0 - p_1$ interval, the pace of technological progress will continue to grow, but slower and slower.

Various factors and effects can be included in the proposed model if, on the one hand, $V(p)$ potential function is replaced by functions of higher orders, and on the other hand, multipliers that take into account the influence of resources involved in scientific research (R&D) sector are introduced into right-hand side of the equation. This modification of the model allows to bring it even closer to reality, but significantly complicates statistical analysis of the effects of unevenness of technological progress.

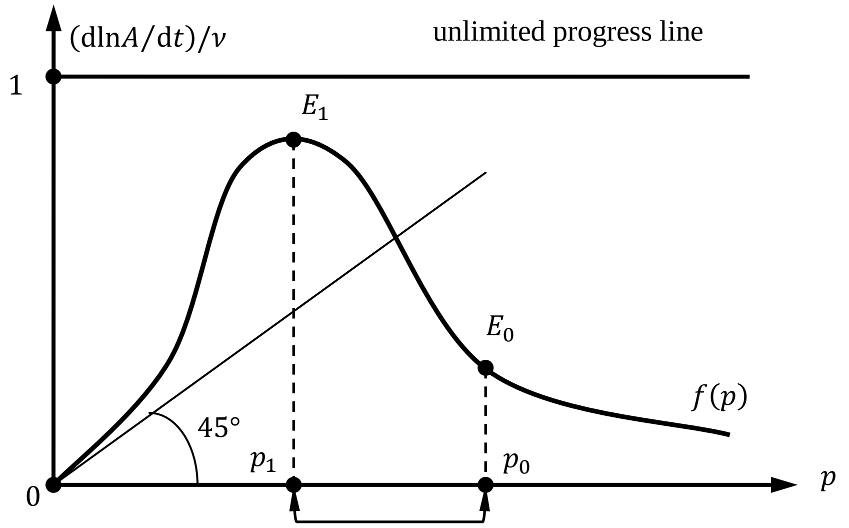

Figure 1. Long-term curve of the pace of technological progress $f(p)$ at $\lambda = 1$.
The $p_0 - p_1$ segment is the time of slow acceleration of technological progress
Source: Zagoruiko (2020)

According to the equation of the dynamics of $V(p)$ function, neither the pace of technological progress nor its acceleration is constant:

$$\frac{\lambda}{\nu} \cdot \frac{\mathrm{d}}{\mathrm{dt}} \ln A = \left(1 - e^{-\lambda p}\right) \cdot \frac{A}{A_{max}}, \quad (17)$$

$$\frac{\lambda}{\nu} \cdot \frac{\mathrm{d}^2}{\mathrm{dt}^2} \ln A = \frac{\mathrm{d}A}{\mathrm{dt}} \cdot \left(\frac{1}{A_{max}} - e^{-\lambda p}\left(\frac{1}{A_{max}} + \frac{1}{A}\right)\right). \quad (18)$$

This means that in the econometric model, long-term curve has to be represented as a result of "technology jumps" - successive shifts of the short-term production function $Y = F(A, K, L)$ due to a change in the efficiency index A.

3.1.2. Choice of a form of production function

For further research, it is important to find out economic content of parameters of the main production functions. Let's start this analysis from elementary production functions - the function with fixed proportions (Leontief function) and linear function:

$$Y = A_K K = A_L L, \qquad Y = A_K K + A_L L, \quad (19), (20)$$

where Y is production; K, L are the values of physical capital and labor involved; A_K, A_L are function parameters.

Let's consider the dimensions of A_K and A_L coefficients. For the Leontief function, they will be determined directly from the function itself:

$$\dim A_K = \dim\left(\frac{Y}{K}\right) = \left(\frac{q}{t}\right) \div \kappa, \qquad \dim A_L = \dim\left(\frac{Y}{L}\right) = \left(\frac{q}{t}\right) \div l, \quad (21), (22)$$

where q is the unit of product measurement, t is time, κ is capital, and l is labor. As we can see, in its economic content, efficiency coefficients A_K and A_L are the performances of production factors. When substituting units of measurement to the Leontief function, the dimensions of left and right parts of the equation coincide:

$$\frac{q}{t} = \frac{q}{\kappa t} \cdot \kappa, \qquad \frac{q}{t} = \frac{q}{lt} \cdot l\,. \quad (23), (24)$$

In this function, A_K and A_L coefficients are mean products according to capital and labor:

$$A_K = \frac{Y}{K}, \qquad A_L = \frac{Y}{L}. \tag{25), (26}$$

In linear function, A_K and A_L coefficients will have the same dimension as in the Leontief function. When substituting their units to linear function, we get:

$$\frac{q}{t} = \frac{q}{\kappa t} \cdot \kappa + \frac{q}{lt} \cdot l\,. \tag{27}$$

However, unlike the Leontief function, each of these coefficients is a mean product of one production factor in the absence of the other one:

$$A_K|_{L=0} = \frac{Y}{K}, \qquad A_L|_{K=0} = \frac{Y}{L}. \tag{28), (29}$$

For both of these functions, the elasticity of replacement of production factors σ is constant - zero for the Leontief function and endless - for linear function.

A function with a constant elasticity of substitution (CES) of production factors, which we will present in the form of:

$$Y^{-\rho/\delta} = (A_K K)^{-\rho} + (A_L L)^{-\rho}, \qquad \rho = 1/\sigma - 1, \tag{30}$$

where ρ and δ are dimensionless function parameters, is the generalization of these functions.

Since $0 \le \sigma \le \infty$, then $-1 \le \rho \le \infty$. The δ parameter is interpreted as an indicator of scale of production. It is logical to believe that in this, more general function, the dimensions of A_K and A_L coefficients are the same as in two previous functions. However, this is only possible provided a permanent scale of production. Indeed, after substitution of units of measurement to the CES function we get:

$$\left(\frac{q}{t}\right)^{-\rho/\delta} = \left(\frac{q}{\kappa t} \cdot \kappa\right)^{-\rho} + \left(\frac{q}{lt} \cdot l\right)^{-\rho}, \tag{31}$$

that is only at $\delta = 1$. In this case A_K and A_L coefficients will be the mean product of one factor of production provided the infinite volume of the other:

$$A_K|_{L=\infty} = \frac{Y}{K}, \qquad A_L|_{K=\infty} = \frac{Y}{L}. \tag{32), (33}$$

Since the Cobb–Douglas function is a boundary case of CES function at $\sigma = 1$, then for it the effect of scale of production should be constant. Let's write it in a form similar to the CES function:

$$Y = (A_K K)^{\beta_K} (A_L L)^{\beta_L}, \tag{34}$$

where β_K, β_L are dimensionless constants, which in their content are the corresponding coefficients of output elasticity. After substitution to this equation of units of measurement, we get:

$$\frac{q}{t} = \left(\frac{q}{\kappa t} \cdot \kappa\right)^{\beta_K} \left(\frac{q}{lt} \cdot l\right)^{\beta_L}. \tag{35}$$

The dimensions of both parts of the equation will only coincide, provided $\beta_K + \beta_L = 1$, that is, a sustainable production effect. With a constant scale effect, each of two A_K and A_L coefficients will be the mean product of the appropriate factor, provided that the other coefficient will also be the mean product:

$$A_K|_{A_L=\frac{Y}{L}} = \frac{Y}{K}, \qquad A_L|_{A_K=\frac{Y}{K}} = \frac{Y}{L}. \tag{36), (37}$$

Therefore, in all the functions considered, A_K and A_L coefficients will characterize the performance of production factors. It is logical to believe that their growth reflects the improvement of the quality of production factors, that is, technological progress. However, unlike previous functions, these coefficients cannot be determined separately in the Cobb–Douglas function. Their geometric mean forms an indicator known as total factor productivity:

$$(A_K)^\beta (A_L)^{1-\beta} = A, \qquad 0 < \beta < 1, \tag{38}$$

where β is the coefficient of output elasticity of capital. The value and dimension of this coefficient will depend on the value of β parameter. However, if A_K and A_L coefficients increase, the aggregate factor productivity will grow, regardless of β changes. This allows to interpret its growth rate

$$\frac{\mathrm{d}}{\mathrm{dt}} \ln A = \frac{\mathrm{d}A}{\mathrm{dt}} \div A \tag{39}$$

as a general pace of technological progress.

The possibility of this rate determination is a significant advantage of the Cobb–Douglas function. Based on these reasons, we will use this function as a tool for researching long-term technology changes.

To reduce the number of variables, the Cobb–Douglas function is traditionally represented in relative values:

$$\frac{Y}{L} = A \cdot \left(\frac{K}{L}\right)^\beta, \quad y = \frac{Y}{L}, \quad k = \frac{K}{L} \quad \Rightarrow, \quad y = A \cdot k^\beta, \qquad (40), (41), (42), (43)$$

where y is labor productivity, k is capital-labor ratio, β is the elasticity of labor productivity by capital-labor ratio. Logarithating this function, we get a linear equation:

$$\ln y = \ln A + \beta \cdot \ln k. \tag{44}$$

There are two independent variables in this equation - $\ln A$ and $\ln k$. In the case of continuous quantities, this equation can be differentiated:

$$\frac{\mathrm{d}}{\mathrm{dt}} \ln y = \frac{\mathrm{d}}{\mathrm{dt}} \ln A + \beta \cdot \frac{\mathrm{d}}{\mathrm{dt}} \ln k. \tag{45}$$

In this equation, d/dt derivatives from a physical point of view are instantaneous velocities, and logarithmic $\mathrm{d}\ln/\mathrm{dt}$ derivatives from an economic point of view are instantaneous growth rates:

$$\frac{\mathrm{d}}{\mathrm{dt}} \ln y = \frac{\mathrm{d}y}{\mathrm{dt}} \div y, \quad \frac{\mathrm{d}}{\mathrm{dt}} \ln A = \frac{\mathrm{d}A}{\mathrm{dt}} \div A, \quad \frac{\mathrm{d}}{\mathrm{dt}} \ln k = \frac{\mathrm{d}k}{\mathrm{dt}} \div k. \qquad (46), (47), (48)$$

Repeated differentiation of logarithmic form of the Cobb–Douglas function leads to equation in acceleration relative to logarithmic quantities:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2} \ln y = \frac{\mathrm{d}^2}{\mathrm{dt}^2} \ln A + \beta \cdot \frac{\mathrm{d}^2}{\mathrm{dt}^2} \ln k. \tag{49}$$

From an economic point of view, $\mathrm{d}^2\ln/\mathrm{dt}^2$ are changes in growth rates.

As a rule, the concepts of speed and acceleration are sufficient to analyze complex natural and economic processes. Higher order derivatives are much more difficult to interpret and use.

3.1.3. Choice of a continuous model of short-term dynamics

Based on the first and second derivatives of the Cobb–Douglas function, two models of short-term dynamics can be obtained.

Thus, if we assume that the pace of technological progress is constant, then we will get a short-term first-order dynamics:

$$\frac{\mathrm{d}}{\mathrm{dt}} \ln A = const, \quad \frac{\mathrm{d}}{\mathrm{dt}} \ln y = \upsilon + \beta \frac{\mathrm{d}}{\mathrm{dt}} \ln k \quad \Rightarrow, \quad y(t) = A(0) \cdot e^{\upsilon t} \cdot k^\beta(t), \qquad (50)\ (51)$$

where υ is the pace of technological progress, which is a positive value ($\upsilon > 0$) with t^{-1} dimension. In the case of continuation of the graph of the $A(t) = A(0)e^{\upsilon t}$ function in the infinitely distant past, A will set up to 0, and in the case of continuation in the infinitely distant future - to infinity:

$$t \longrightarrow -\infty \quad \Rightarrow \quad A(0)e^{\upsilon t} \longrightarrow 0, \tag{52}$$

$$t \longrightarrow +\infty \quad \Rightarrow \quad A(0)e^{\upsilon t} \longrightarrow +\infty. \tag{53}$$

Assuming that the acceleration of technological progress is a constant ($\dim \alpha = t^{-1}$), then we will get a second-order short-term dynamic model:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\ln A = const \quad \Rightarrow \quad \frac{\mathrm{d}^2}{\mathrm{dt}^2}\ln y = \alpha + \beta \frac{\mathrm{d}^2}{\mathrm{dt}^2}\ln k \quad \Rightarrow \tag{54}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\ln y = \upsilon(0) + \alpha t + \beta \frac{\mathrm{d}}{\mathrm{dt}}\ln k. \tag{55}$$

In the point $t = -\upsilon(0)/\alpha$ the function

$$A(t) = A(0) \cdot \exp\left(\upsilon(0)t + \alpha \frac{t^2}{2}\right) \tag{56}$$

reaches the extremum. In the case of $\alpha > 0$ it will be a minimum point, and in the case of $\alpha < 0$ it will be a maximum one.

The change of υ parameter in dynamic first-order model can be interpreted as a noticeable change in the angle of inclination of long-term curve of technological progress. The change of α parameter in dynamic second-order model can be interpreted as a noticeable change in concavity (or convexity) of a long-term curve. This type of change is more important from a theoretical point of view. It allows to directly find the possible effect of slowing down of technological progress. Thus, in the case of continuous quantities, for this study a second-order short-term model is more suitable.

3.2. Discrete models of short-term dynamics

3.2.1. Choice of a discrete model of short-term dynamics

To statistically check the original theoretical model, it must be written in discrete values. With this transition, the original (static) form of the Cobb–Douglas function will only be stored if the derivatives under consideration will be replaced by the corresponding differences of the first and second orders:

$$\Delta \ln y = \Delta \ln A + \beta \cdot \Delta \ln k, \tag{57}$$

$$\Delta^{(2)} \ln y = \Delta^{(2)} \ln A + \beta \cdot \Delta^{(2)} \ln k. \tag{58}$$

By analogy with differential quantities, logarithmic growth in the first equation will be interpreted as annual growth rates:

$$\Delta_{\mathrm{t}} \ln \mathrm{y} = \ln \mathrm{y_t} - \ln \mathrm{y_{t-1}} = \ln\left(\frac{\mathrm{y_t}}{\mathrm{y_{t-1}}}\right), \tag{59}$$

where $\mathrm{y_t}/\mathrm{y_{t-1}}$ is the rate (index number) of labor productivity for t year (and similar to k and A). The ratio of two such indices can logically be called the second-order index, and their logarithm is the rate of second-order increase (acceleration rate):

$$\ln\left(\frac{y_t}{y_{t-1}} \div \frac{y_{t-1}}{y_{t-2}}\right) = \ln\left(\frac{y_t}{y_{t-1}}\right) - \ln\left(\frac{y_{t-1}}{y_{t-2}}\right) = \Delta_t \ln y - \Delta_{t-1} \ln y = \Delta_t^{(2)} \ln y. \tag{60}$$

It is clear that direct use of the original (static) form of the Cobb–Douglas function is likely to lead to autocorrelation in the ranks of the observed y and k quantities. The use of the dynamics equation instead eliminates autocorrelation, which corresponds to the first-order auto-regulation scheme.

The first theoretical model

$$\Delta \ln y = \upsilon + \beta \cdot \Delta \ln k \tag{61}$$

describes the first-order short-term output dynamics (STOD) function. The second theoretical model

$$\Delta^{(2)} \ln y = \alpha + \beta \cdot \Delta^{(2)} \ln k \tag{62}$$

describes the second-order STOD function.

Graphs of both functions may shift over time. Thus, if υ and α parameters change at constant β value, then the graphs of these functions will shift in parallel to the previous ones. If, on the contrary, β parameter by unchanged υ and α will change, then the graph of new STOD function will be located on new ray that will come from the same point on the ordinate as the previous one.

In the case of the use of the second dynamic model, the number of short-term second-order equations will be less than the number of such first-order equations, which simplifies the analysis of the results obtained. However, the main advantage of this model is as follows. It directly contains α - the rate of acceleration of technological progress. If it turns out that in each subsequent period this parameter decreases, then it can be considered a significant argument in favor of a hypothesis of limited technological progress

Thus, in the case of discrete quantities, a short-term second-order model is also more acceptable.

3.2.2. Econometric model of shifts of the second-order STOD function

3.2.2.1. Periodization criteria

Let's introduce the following designations of the accelerations:

$$\Delta^{(2)} \ln y = z, \qquad \Delta^{(2)} \ln k = x, \qquad \Delta^{(2)} \ln A = \alpha. \qquad (63), (64), (65)$$

Then generalized models of linear regression for individual short-term periods will take the following form:

$$z_{Tt} = \alpha_T + \beta_T x_{Tt} + \varepsilon_{Tt}, \qquad T = 1, \dots, N, \qquad t = 1, \dots, n_T, \qquad (66)$$

where T, t are the number of a short-term period and the number of the year during this period; N, n_T are the number of short-term periods and the number of years in T period; z_{Tt}, x_{Tt} are the pace of acceleration of productivity and capital-labor ratio in Tt year; α_T, β_T are the pace of technological acceleration and the elasticity ratio in T period; ε_{Tt} is an unobserved random value in Tt year.

In order to construct the equations of STOD functions according to the sample data, it is necessary to divide the long-term observation period into separate short-term ones.

According to the hypothesis of sustainable effect of production scale, the elasticity coefficient β should be less than one. On the other hand, only positive relationship between factors of production and the output corresponds to the content of the production function. Therefore, this parameter should be within:

$$0 < \beta < 1. \qquad (67)$$

This restriction will be used as a criterion for the separation of short-term periods. It is logical to consider the years, in which the condition

$$0 < \frac{\Delta^{(3)} y}{\Delta^{(3)} k} = \frac{z_{T,t} - z_{T,t-1}}{x_{T,t} - x_{T,t-1}} < 1 \qquad (68)$$

will be disturbed, as the moments of a shift in the STOD function.

Thus, the equations:

$$z_T = a_T + b_T x_T + e_T, \qquad T = 1, \dots, N, \qquad (69)$$

will be estimates of models of STOD functions according to sampling,

where a_T, b_T are the estimates of unknown parameters of the STOD function in T period; z_T, x_T and e_T are the vectors of actual values of variables and errors in T period:

$$z_T = \left(z_{T1}, z_{T2}, \dots, z_{Tn_T}\right)^{\mathbb{Tr}}, \quad x_T = \left(x_{T1}, x_{T2}, \dots, x_{Tn_T}\right)^{\mathbb{Tr}}, \qquad (70)\ (71)$$

$$e_T = \left(e_{T1}, e_{T2}, \dots, e_{Tn_T}\right)^{\mathbb{Tr}} \qquad (72)$$

and $(\bullet)^{\mathbb{Tr}}$ is the transposition operation.

To find out the nature of possible shift of the STOD function in T period, one must check the significance of its a_T and b_T parameters. In this case, as hypothetical values we will choose theoretical values of these parameters for the previous period. The corresponding null hypothesis and Student's t-test will look like:

$$H_0(a_T): a_T = a_{T-1}, \qquad t(a_T) = \frac{a_T - a_{T-1}}{\hat{\sigma}(a_T)}, \tag{73), (74}$$

$$H_0(b_T): b_T = b_{T-1}, \qquad t(b_T) = \frac{b_T - b_{T-1}}{\hat{\sigma}(b_T)}, \tag{75), (76}$$

where $\hat{\sigma}(a_T)$ and $\hat{\sigma}(b_T)$ are standard errors of regression coefficient.

If it turns out, for example, that $H_0(b_T)$ hypothesis should be accepted, and $H_0(a_T)$ hypothesis should be rejected, then it is possible to conclude about a parallel shift of the graph of the STOD function in T period compared to the previous one, that is, about a change in the pace of acceleration of technological progress at a constant output elasticity. Since in this case both values of b parameter are statistically equivalent, it should be recalculated using data for both periods. As a result, the STOD function for this period will take the form:

$$\hat{z}_{Tt} = a_T + b_{T,T-1} x_{Tt}, \tag{77}$$

where $\hat{z}_{Tt}$ is theoretical (predicted) z value in Tt year; $b_{T,T-1}$ is calculated value of the parameter, which is the same for T and $T-1$ periods. Similarly, it will be necessary to recalculate a parameter if $H_0(a_T)$ is accepted.

3.2.2.2. Method of dummy variables

The well-known method of dummy variables is an adequate method for such calculations. Since the values of the parameters in a certain period are compared not with the initial ones, but with the previous ones, we will use dummy variables in all periods, starting with the first one: $D_1, \ldots, D_\tau, \ldots, D_N$. By definition, each of these variables equals 1 in its own short-term period and 0 in the others. As a result, we get the original long-term model:

$$\begin{cases} \hat{z}_{Tt} = \sum_{\tau=1}^{N} (a_\tau + b_\tau x_{Tt}) D_{\tau,Tt} \quad T = 1, \ldots, N \quad t = 1, \ldots, n_T \\ \tau = T \quad \Rightarrow \quad D_{\tau,Tt} = 1 \\ \tau \neq T \quad \Rightarrow \quad D_{\tau,Tt} = 0 \end{cases}, \tag{78}$$

in which both parameters will have different values in each short-term period.

Let's write this model in vector-matrix form:

$$z = X \binom{a}{b} + e, \tag{79}$$

where $z = (z_1, z_2, \ldots, z_N)^{\mathbb{Tr}}$, $e = (e_1, e_2, \ldots, e_N)^{\mathbb{Tr}}$ are vectors, the components of which are vectors of the corresponding periods; $a = (a_1, a_2, \ldots, a_N)^{\mathbb{Tr}}$, $b = (b_1, b_2, \ldots, b_N)^{\mathbb{Tr}}$ are vectors, the components of which are the values of the parameters in the corresponding period; X s a matrix of the form:

$$X = \left(\begin{array}{cccc|cccc} (1) & (0) & \ldots & (0) & (x_{1t}) & (0) & \ldots & (0) \\ (0) & (1) & \ldots & (0) & (0) & (x_{2t}) & \ldots & (0) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (0) & (0) & \ldots & (1) & (0) & (0) & \ldots & (x_{Nt}) \end{array} \right). \tag{80}$$

Column vectors, namely: $(1) = (1,1\ldots1)^{\mathbb{Tr}}$, $(0) = (0,0\ldots0)^{\mathbb{Tr}}$, $(x_{Tt}) = \left(x_{T1}, x_{T2} \ldots x_{Tn_T}\right)^{\mathbb{Tr}}$ are the elements of this matrix. The number of elements in each such vector is the number of years in the corresponding short-term period. Thus, all vectors of one row of X matrix have the same size and correspond to one short-term period. With respect to these vectors, both halves of X matrix are square matrices. The size of the halves of X matrix is determined by the number of N short-term periods.

The objective function of the proposed model is determined by the least square method:

$$\sum_{T=1}^{N}\sum_{t=1}^{n_T} e_{Tt}^2 = e^{\mathbb{Tr}}e = \left[z - X\begin{pmatrix}a\\b\end{pmatrix}\right]^{\mathbb{Tr}}\left[z - X\begin{pmatrix}a\\b\end{pmatrix}\right] \to min. \quad (81)$$

Hence, the system of normal equations and its solution will have the form:

$$(X^{\mathbb{Tr}}X)\begin{pmatrix}a\\b\end{pmatrix} = X^{\mathbb{Tr}}z \qquad \Rightarrow \qquad \begin{pmatrix}a\\b\end{pmatrix} = (X^{\mathbb{Tr}}X)^{-1}X^{\mathbb{Tr}}z. \quad (82), (83)$$

The transposed matrix will have the form:

$$X^{\mathbb{Tr}} = \left(\begin{array}{cccc} (1)^{\mathbb{Tr}} & (0)^{\mathbb{Tr}} & \dots & (0)^{\mathbb{Tr}} \\ (0)^{\mathbb{Tr}} & (1)^{\mathbb{Tr}} & \dots & (0)^{\mathbb{Tr}} \\ \dots & \dots & \dots & \dots \\ (0)^{\mathbb{Tr}} & (0)^{\mathbb{Tr}} & \dots & (1)^{\mathbb{Tr}} \\ \hline (x_{1t})^{\mathbb{Tr}} & (0)^{\mathbb{Tr}} & \dots & (0)^{\mathbb{Tr}} \\ (0)^{\mathbb{Tr}} & (x_{2t})^{\mathbb{Tr}} & \dots & (0)^{\mathbb{Tr}} \\ \dots & \dots & \dots & \dots \\ (0)^{\mathbb{Tr}} & (0)^{\mathbb{Tr}} & \dots & (x_{Nt})^{\mathbb{Tr}} \end{array}\right). \quad (84)$$

The $X^{\mathbb{Tr}}X$ product is a square symmetric matrix consisting of diagonal blocks:

$$X^{\mathbb{Tr}}X = \begin{pmatrix} \mathrm{diag}(n_T) & \mathrm{diag}\left(\sum_{t=1}^{n_T} x_{Tt}\right) \\ \mathrm{diag}\left(\sum_{t=1}^{n_T} x_{Tt}\right) & \mathrm{diag}\left(\sum_{t=1}^{n_T} x_{Tt}^2\right) \end{pmatrix}. \quad (85)$$

The numbers of years in short-term periods are along the main diagonal of the upper left block, the sums of squares of the independent variable in short-term periods are along the main diagonal of the lower right block. The sums of the first powers of the independent variable are located along the main diagonals of the other two blocks.

The vector of free members consists of two blocks:

$$X^{\mathbb{Tr}}z = \begin{pmatrix} \left(\sum_{t=1}^{n_T} z_{Tt}\right) \\ \left(\sum_{t=1}^{n_T} x_{Tt}z_{Tt}\right) \end{pmatrix}. \quad (86)$$

Its upper block contains the sums of the first powers of the dependent variable, and the lower block contains the sums of the products of the independent and dependent variables.

3.2.2.3. Initial stage of calculations

At the initial stage of calculations, the parameters of the proposed model will be defined as:

$$\begin{pmatrix}a_T\\b_T\end{pmatrix} = \frac{1}{\det(X_T^{\mathbb{Tr}}X_T)} \cdot \begin{pmatrix} \sum_t x_{Tt}^2 & -\sum_t x_{Tt} \\ -\sum_t x_{Tt} & n_T \end{pmatrix} \cdot \begin{pmatrix} \sum_t z_{Tt} \\ \sum_t x_{Tt}z_{Tt} \end{pmatrix}, \quad T = 1,\dots,N, \quad (87)$$

where $X_T = ((1) \quad (x_{Tt}))$ is a block of the general X matrix that describes only T period. In X_T matrix, the left block is a column vector, which consists of units, and the right block is a column vector, which consists of the values of x variable in T period. The determinant of $X_T^{\mathbb{Tr}}X_T$ product will be equal to:

$$\det(X_T^{\mathbb{Tr}}X_T) = \begin{vmatrix} n_T & \sum_t x_{Tt} \\ \sum_t x_{Tt} & \sum_t x_{Tt}^2 \end{vmatrix} = n_T\sum_t x_{Tt}^2 - \left(\sum_t x_{Tt}\right)^2. \quad (88)$$

Thus, all a_T and b_T parameters will initially be the same as in the model without dummy variables. However, already at this stage, standard error of regression equation for the long-term model will differ from the similar value for the short-term model:

$$\hat{\sigma}(e) = \sqrt{\frac{\sum_{T=1}^{N}\sum_{t=1}^{n_T} e_{Tt}^2}{\sum_{T=1}^{N} n_T - 2N}}. \quad (89)$$

As a result, standard errors of regression coefficients for T period will also differ:

$$\hat{\sigma}(a_T) = \hat{\sigma}(e)\sqrt{\frac{n_T \sum_t x_{Tt}^2}{\det(X_T^{\mathbb{Tr}} X_T)}}, \qquad \hat{\sigma}(b_T) = \hat{\sigma}(e)\sqrt{\frac{n_T^2}{\det(X_T^{\mathbb{Tr}} X_T)}}. \quad (90), (91)$$

3.2.2.4. Case of a change in output elasticity

Let's consider the case when $H_0(a_T): a_T = a_{T-1}$ hypothesis is accepted, and $H_0(b_T): b_T = b_{T-1}$ hypothesis is rejected. This means that in the block of X matrix describing these two periods, two left columns must be summed:

$$X_{T-1,T} = \left(\begin{matrix}(1) & (0)\\ (0) & (1)\end{matrix}\middle|\begin{matrix}(x_{T-1,t}) & (0)\\ (0) & (x_{Tt})\end{matrix}\right) \Rightarrow X_{a(T-1,T)} = \left(\begin{matrix}(1)\\ (1)\end{matrix}\middle|\begin{matrix}(x_{T-1,t}) & (0)\\ (0) & (x_{Tt})\end{matrix}\right). \quad (92)$$

From here we get the equation for calculating the parameters in $T-1$ and T periods:

$$\begin{pmatrix} a\\ b\end{pmatrix} = (X_{Ua}^{\mathbb{Tr}} X_{Ua})^{-1} X_{Ua}^{\mathbb{Tr}} z_{T-1,T}, \quad (93)$$

where

$$X_{Ua} = X_{a(T-1,T)}, \qquad z_{T-1,T} = \begin{pmatrix}(z_{T-1,t})\\ (z_{Tt})\end{pmatrix}, \quad (94), (95)$$

or in expanded form

$$\begin{pmatrix} a_{T,T-1}\\ b_{T-1}\\ b_T\end{pmatrix} = \begin{pmatrix} n_{T-1}+n_T & \sum_t x_{T-1,t} & \sum_t x_{Tt}\\ \sum_t x_{T-1,t} & \sum_t x_{T-1,t}^2 & 0\\ \sum_t x_{Tt} & 0 & \sum_t x_{Tt}^2\end{pmatrix}^{-1} \cdot \begin{pmatrix}\sum_t z_{T-1,t} + \sum_t z_{Tt}\\ \sum_t x_{T-1,t} z_{T-1,t}\\ \sum_t x_{Tt} z_{Tt}\end{pmatrix}, \quad (96)$$

where $a_{T,T-1}$ is the value of a parameter, which is the same for these two periods.

In a model with new parameters, both the sum of errors and standard error of regression equation will change:

$$\hat{\sigma}(e) = \sqrt{\frac{\sum_{\theta=1}^{T-2}\sum_{t=1}^{n_\theta} e_{\theta t}^2 + \sum_{t=1}^{n_{T-1}} e_{T-1,t}^2 + \sum_{t=1}^{n_T} e_{T,t}^2 + \sum_{\theta=T+1}^{N}\sum_{t=1}^{n_\theta} e_{\theta t}^2}{\sum_{T=1}^{N} n_T - 2N + 1}}. \quad (97)$$

The determinant of $X_{Ua}^{\mathbb{Tr}} X_{Ua}$ matrix will be equal to

$$\det(X_{Ua}^{\mathbb{Tr}} X_{Ua}) = \sum_t x_{Tt}\begin{vmatrix}\sum_t x_{T-1,t} & \sum_t x_{T-1,t}^2\\ \sum_t x_{Tt} & 0\end{vmatrix} + \sum_t x_{Tt}^2\begin{vmatrix} n_{T-1}+n_T & \sum_t x_{T-1,t}\\ \sum_t x_{T-1,t} & \sum_t x_{T-1,t}^2\end{vmatrix}. \quad (98)$$

Let's denote the elements of $\det(X_{Ua}^{\mathbb{Tr}} X_{Ua})(X_{Ua}^{\mathbb{Tr}} X_{Ua})^{-1}$ matrix as (x_{aij}) and define them using algebraic complements:

$$x_{a11} = (-1)^{1+1}\begin{vmatrix}\sum_t x_{T-1,t}^2 & 0\\ 0 & \sum_t x_{Tt}^2\end{vmatrix}, \qquad x_{a12} = x_{a21} = -\sum_t x_{T-1,t}\sum_t x_{Tt}^2, \quad (99), (100)$$

$$x_{a22} = (-1)^{2+2}\begin{vmatrix} n_{T-1}+n_T & \sum_t x_{Tt}\\ \sum_t x_{Tt} & \sum_t x_{Tt}^2\end{vmatrix}, \qquad x_{a23} = x_{a32} = \sum_t x_{T-1,t}\sum_t x_{Tt}, \quad (101), (102)$$

$$x_{a33} = (-1)^{3+3}\begin{vmatrix} n_{T-1}+n_T & \sum_t x_{T-1,t}\\ \sum_t x_{T-1,t} & \sum_t x_{T-1,t}^2\end{vmatrix}, \qquad x_{a13} = x_{a31} = -\sum_t x_{T-1,t}^2\sum_t x_{Tt}. \quad (103), (104)$$

Standard errors of new regression coefficients will be equal to $\hat{\sigma}(e)$ product by the square root of diagonal elements of $(X_{Ua}^{\mathbb{Tr}} X_{Ua})^{-1}$ matrix, namely:

$$\hat{\sigma}(a_{T,T-1}) = \hat{\sigma}(e)\sqrt{\frac{x_{a11}}{\det(X_{Ua}^{\mathbb{Tr}} X_{Ua})}}, \quad (105)$$

$$\hat{\sigma}(b_{T-1}) = \hat{\sigma}(e)\sqrt{\frac{x_{a22}}{\det(X_{Ua}^{\mathbb{Tr}} X_{Ua})}}, \quad (106)$$

$$\hat{\sigma}(b_T) = \hat{\sigma}(e)\sqrt{\frac{x_{a33}}{\det(X_{Ua}^{\mathbb{Tr}} X_{Ua})}}. \quad (107)$$

3.2.2.5. Case of changing the pace of acceleration of technological progress

Let's consider the opposite case, when $H_0(b_T)\colon b_T = b_{T-1}$ hypothesis is accepted, and $H_0(a_T)\colon a_T = a_{T-1}$ hypothesis is rejected. This means that in the block of X matrix describing these two periods, two right columns must be summed:

$$X_{T,T-1} = \left(\begin{matrix}(1) & (0)\\(0) & (1)\end{matrix}\middle|\begin{matrix}(x_{T-1,t}) & (0)\\(0) & (x_{Tt})\end{matrix}\right) \Rightarrow \quad X_{b(T-1,T)} = \left(\begin{matrix}(1) & (0)\\(0) & (1)\end{matrix}\middle|\begin{matrix}(x_{T-1,t})\\(x_{Tt})\end{matrix}\right). \quad (108)$$

From here we get the equation for calculating the parameters in $T-1$ and T periods:

$$\begin{pmatrix}a\\b\end{pmatrix} = \left(X_{Ub}^{\mathbb{Tr}} X_{Ub}\right)^{-1} X_{Ub}^{\mathbb{Tr}} z_{T-1,T}, \quad (109)$$

where

$$X_{Ub} = X_{b(T-1,T)}, \qquad z_{T-1,T} = \begin{pmatrix}(z_{T-1,t})\\(z_{Tt})\end{pmatrix}, \quad (110), (111)$$

or in expanded form

$$\begin{pmatrix}a_{T-1}\\a_T\\b_{T,T-1}\end{pmatrix} =$$

$$= \begin{pmatrix}n_{T-1} & 0 & \sum_t x_{T-1,t}\\0 & n_T & \sum_t x_{Tt}\\\sum_t x_{T-1,t} & \sum_t x_{Tt} & \sum_t x_{T-1,t}^2 + \sum_t x_{Tt}^2\end{pmatrix}^{-1} \cdot \begin{pmatrix}\sum_t z_{T-1,t}\\\sum_t z_{T,t}\\\sum_t x_{T-1,t} z_{T-1,t} + \sum_t x_{Tt} z_{Tt}\end{pmatrix}, \quad (112)$$

where $b_{T,T-1}$ is the value of b parameter, which is the same for these two periods.

In this version of the model, standard error of regression equation will be calculated using the same formula as in the previous case.

The determinant of $X_{Ub}^{\mathbb{Tr}} X_{Ub}$ matrix will be equal to

$$\det\left(X_{Ub}^{\mathbb{Tr}} X_{Ub}\right) = n_{T-1}\begin{vmatrix}n_T & \sum_t x_{Tt}\\\sum_t x_{Tt} & \sum_t x_{T-1,t}^2 + \sum_t x_{Tt}^2\end{vmatrix} + \sum_t x_{T-1,t}\begin{vmatrix}0 & \sum_t x_{T-1,t}\\n_T & \sum_t x_{Tt}\end{vmatrix}. \quad (113)$$

Let's denote the elements of $\det\left(X_{Ub}^{\mathbb{Tr}} X_{Ub}\right)\left(X_{Ub}^{\mathbb{Tr}} X_{Ub}\right)^{-1}$ matrix as (x_{bij}) and define them using algebraic complements:

$$x_{b11} = (-1)^{1+1}\begin{vmatrix}n_T & \sum_t x_{Tt}\\\sum_t x_{Tt} & \sum_t x_{T-1,t}^2 + \sum_t x_{Tt}^2\end{vmatrix}, \quad x_{b12} = x_{b21} = \sum_t x_{Tt} \sum_t x_{T-1,t}, \quad (114), (115)$$

$$x_{b22} = (-1)^{2+2}\begin{vmatrix}n_{T-1} & \sum_t x_{T-1,t}\\\sum_t x_{T-1,t} & \sum_t x_{T-1,t}^2 + \sum_t x_{Tt}^2\end{vmatrix}, \quad x_{b23} = x_{b32} = -n_{T-1}\sum_t x_{Tt}, \quad (116), (117)$$

$$x_{b33} = (-1)^{3+3}\begin{vmatrix}n_{T-1} & 0\\0 & n_T\end{vmatrix} \qquad x_{b13} = x_{b31} = -n_T \sum_t x_{T-1,t}. \quad (118), (119)$$

Standard errors of new regression coefficients will be equal to the product of $\hat{\sigma}(e)$ by the square root of diagonal elements of $\left(X_{Ub}^{\mathbb{Tr}} X_{Ub}\right)^{-1}$ matrix.

3.2.2.6. Change in output elasticity and a subsequent change in the pace of progress acceleration

In addition to the considered cases, more complex ones are also possible. In particular, if $H_0(a_T)\colon a_T = a_{T-1}$ and $H_0(b_{T+1})\colon b_{T+1} = b_T$ hypotheses are accepted and $H_0(a_{T+1})\colon a_{T+1} = a_T$ and $H_0(b_T)\colon b_T = b_{T-1}$ hypotheses are rejected, parameter vector will consist of components:

$$(a \quad b)^{\mathbb{Tr}} = (a_{T,T-1} \quad a_{T+1} \quad b_{T-1} \quad b_{T+1,T})^{\mathbb{Tr}}. \tag{120}$$

The block of X matrix, which describes $T-1$, T, $T+1$ periods, will take the form:

$$X_{a(T-1,T),b(T,T+1)} = \left(\begin{array}{cc|cc} (1) & (0) & (x_{T-1,t}) & (0) \\ (1) & (0) & (0) & (x_{Tt}) \\ (0) & (1) & (0) & (x_{T+1,t}) \end{array}\right). \tag{121}$$

A column of free members will correspond to it:

$$X^{\mathbb{Tr}}_{a(T-1,T),b(T,T+1)} \cdot \begin{pmatrix} (z_{T-1,t}) \\ (z_{Tt}) \\ (z_{T+1,t}) \end{pmatrix} = \begin{pmatrix} \sum_t z_{T-1,t} + \sum_t z_{Tt} \\ \sum_t z_{T+1,t} \\ \sum_t x_{T-1,t} z_{T-1,t} \\ \sum_t x_{Tt} z_{Tt} + \sum_t x_{T+1,t} z_{T+1,t} \end{pmatrix}. \tag{122}$$

3.2.2.7. Generalization of the model for the case of several studied countries

The model considered above can be generalized for the case of several studied countries. This generalization can be done in several ways.

The first approach is that short-run production functions are considered exclusively at the national level. According to this approach, the world economy is presented as a simple sum of national economies. According to this idea, a conclusion about the nature of technological progress is made on the basis of the mean or the maximum national values of acceleration pace:

$$\overline{a_\vartheta} = \frac{1}{M} \cdot \sum_{m=1}^{M} a_{m\vartheta}, \qquad a_{\max\vartheta} = \max\{a_{1\vartheta}, a_{2\vartheta}, \dots a_{M\vartheta}\}, \tag{123), (124}$$

where ϑ is the calendar year number; m, M are the country number and the number of countries under study; a is the pace of acceleration of technological progress.

According to the second approach, an international production function is introduced, in which the variables are the mean values of the corresponding national values:

$$Y = F\left(\overline{K_\vartheta}, \overline{L_\vartheta}\right). \tag{125}$$

At the same time, both the arithmetic mean and the geometric mean values can be chosen as the mean values. Both types of the mean values can be simple or weighted ones.

So, for capital, the simple mean values will have the form:

$$A(K_\vartheta) = \frac{1}{M} \cdot \sum_{m=1}^{M} K_{m\vartheta}, \qquad G(K_\vartheta) = \prod_{m=1}^{M} (K_{m\vartheta})^{\frac{1}{M}}. \tag{126), (127}$$

In the formulas of the weighted mean values, the share of the indicator of a certain country in the sum of national indicators of all countries is chosen as coefficients:

$$\kappa_{m\vartheta} = \frac{K_{m\vartheta}}{\sum_{\mu=1}^{M} K_{\mu\vartheta}} \qquad \Longrightarrow, \tag{128}$$

$$WA(K_\vartheta) = \sum_{m=1}^{M} \kappa_{m\vartheta} \cdot K_{m\vartheta}, \qquad WG(K_\vartheta) = \prod_{m=1}^{M} (K_{m\vartheta})^{\kappa_{m\vartheta}}. \tag{129), (130}$$

4. CONCLUSIONS AND RECOMMENDATIONS

According to the tasks set in this study, the following results have been obtained.
1. The concept of the potential for increasing the efficiency of production factors is introduced and the exponential function of this potential is proposed. The differential equation, in which the potential function decreases at a constant rate, does not contradict the history of technological development.
2. The choice of the Cobb-Douglas function as the initial production function is substantiated. On its basis, a linear model of long-term economic growth, consisting of second-order growth equations in successive short-term periods, is built.
3. It is shown that the parameters of short-term equations can be determined by an iterative procedure using the method of dummy variables. At the initial stage of calculations, the years in which the condition of constant effect of production scale is violated are determined. These years are chosen as moments of shift in the short-term function of output dynamics. Next, for the obtained parameters null hypotheses are formulated and Student's t-tests are applied. According to the results of testing these hypotheses, it is determined for which adjacent periods a certain parameter should be the same. After that, the calculations are repeated.
4. The tendency to decrease the pace of technological progress acceleration in developed countries can be interpreted as the result of approaching the limit of efficiency of the world economy.

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Conflict of interest
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ОБМЕЖЕННЯ СВІТОВОГО ТЕХНОЛОГІЧНОГО ПРОГРЕСУ: МЕТОДОЛОГІЯ МОДЕЛЮВАННЯ ТА ПЕРЕВІРКИ

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Анотація. Статтю присвячено моделюванню обмеженого технологічного прогресу та методології перевірки цієї гіпотези.

В основу вихідної теоретичної моделі покладено поняття потенціалу підвищення ефективності факторів виробництва – відношення абсолютного резерву підвищення ефективності до її досягнутого рівня. Цей показник виступає аргументом експоненціальної функції, яка має властивості, аналогічні властивостям самого потенціалу. Елементарною версією моделі обмеженого технологічного прогресу є диференціальне рівняння, в якому функція потенціалу скорочується зі сталим темпом. Розв'язком цього рівняння в системі координат «потенціал – темп технологічного прогресу» є «мегахвиля», що починається у нескінченно далекому минулому і завершується у нескінченно далекому майбутньому. На думку авторів, такий розв'язок не суперечить історії технологічного розвитку.

Як вихідна функція для побудови економетричної моделі обрана функція Кобба–Дугласа. На її основі побудована лінійна модель довгострокового економічного зростання, що складається з рівнянь приростів другого порядку в послідовних короткострокових періодах.

Показано, що параметри короткострокових рівнянь можна визначити за ітеративною процедурою з використанням методу фіктивних змінних. На початковому етапі обчислень визначаються роки, в які порушується умова сталого ефекту масштабу виробництва. Ці роки обираються за моменти зрушення короткострокової функції динаміки випуску. Далі, для отриманих параметрів формулюються нульові гіпотези та застосовуються критерії Стьюдента. Відповідно до результатів перевірки цих гіпотез визначається, для яких суміжних періодів певний параметр має бути однаковим. Після цього розрахунки повторюються. Якщо виявиться, що в кожному наступному періоді темп прискорення технологічного прогресу зменшується, то це можна вважати вагомим аргументом на користь гіпотези його обмеженості.

Запропоновано декілька підходів до узагальнення цієї економетричної моделі на випадок дослідження декількох країн.

Ключові слова: хвилі технологічного прогресу, фактори виробництва, потенціал ефективності, функція Кобба–Дугласа, світова економіка